

HAMILTONIANS AND PHYSICAL VACUA OF EXACTLY SOLVABLE MODELS*

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Abstract

Correct quantum Hamiltonians of a few exactly solvable models in two space-time dimensions are derived by taking into account operator solutions of the field equations. While two versions of the model with derivative-coupling are found to be equivalent in many respects to a free theory, physical vacua of the massless Thirring and Federbush models are obtained by means of a Bogoliubov transformation in the form of a coherent state quadratic in composite boson operators. Contrary to the conventional treatment, the Federbush model is shown to have the same interacting structure in both space-like and light-front formulations.

1 Introduction

Exactly solvable models, i.e. simple relativistic theories in $D=1+1$ in which operator solutions of field equations are known, provide us with a suitable arena for analyzing the nonperturbative structure of the usual ("space-like" - SL) and light-front (LF) forms [1] of hamiltonian field theory and also for a comparison between them. In this paper, we will study models with derivative coupling (DCM) [2, 3, 4], the massless Thirring model (TM) [5] and the massive Federbush model (FM) [6]. The main new idea is to make use of the knowledge of the operator solutions to express the Lagrangian and Hamiltonian entirely in terms of true field degrees of freedom, which are actually free fields. In the case of the TM and FM, the diagonalization of the SL Hamiltonian by means of a Bogoliubov transformation generates the true vacuum, i.e. the lowest-energy eigenstate of the full Hamiltonian, not just of its free part. This vacuum state is a transformed Fock vacuum - a coherent state quadratic in composite boson operators that represent currents.

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2 Derivative-coupling models

The general Lagrangian density of the model with derivative coupling is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}', \quad \mathcal{L}_0 = \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m \bar{\Psi} \Psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \mu^2 \phi^2, \quad \mathcal{L}' = -g \partial_\mu \phi K^\mu. \quad (1)$$

K^μ can be the vector current $J^\mu(x)$ or the axial-vector current $J_5^\mu(x)$. We will briefly analyze the case with J_5^μ and $m = 0$ (the Rothe-Stamatescu (RS) model [3]). The same model with massive fermions is not exactly solvable [4]. The corresponding field equations

$$i\gamma^\mu \partial_\mu \Psi = g \partial_\mu \phi \gamma^\mu \gamma^5 \Psi, \quad \partial_\mu \partial^\mu \phi + \mu^2 \phi = g \partial_\mu J_5^\mu = 0 \quad (2)$$

are solved in terms of the free scalar field $\phi(x)$ and the massless fermion field $\psi(x)$ as

$$\Psi(x) =: e^{-ig\gamma^5\phi(x)} : \psi(x), \quad \psi(x) = \int_{-\infty}^{+\infty} \frac{dp^1}{\sqrt{2\pi}} \{b(p^1)u(p^1)e^{-ip \cdot x} + d^\dagger(p^1)v(p^1)e^{ip \cdot x}\}, \quad (3)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dk^1}{\sqrt{2E(k^1)}} [a(k^1)e^{-i\hat{k} \cdot x} + a^\dagger(k^1)e^{i\hat{k} \cdot x}], \quad \hat{k} \cdot x \equiv E(k^1)t - k^1x^1, \quad (4)$$

$E(k^1) = \sqrt{k_1^2 + \mu^2}$. Since the solution (4) tells us that the interacting fermion field $\Psi(x)$ is composed from two free fields, we should formulate the dynamics of the model in terms of these degrees of freedom by inserting the solution to the Lagrangian. Consequently, the contribution of the derivative term in \mathcal{L}_0 eliminates the interaction term! Thus

$H = H_0 = \int_{-\infty}^{+\infty} dx^1 [-i\psi^\dagger \alpha^1 \partial_1 \psi + \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\partial_1 \phi)^2 + \frac{1}{2}\mu^2 \phi^2]$. Clearly, $|vac\rangle \equiv |0\rangle$. The signature of an interacting theory is g -dependent correlation functions calculated from (3). With $D^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$, $S^{(+)}(x-y) = \langle 0|\psi(x)\bar{\psi}(y)|0\rangle$, we get e.g.

$$\langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle = e^{g^2 D^{(+)}(x-y)} S^{(+)}(x-y). \quad (5)$$

Note that conventionally one replaces $\Psi(x)$ by the free field in \mathcal{L}_0 . This procedure yields

$$H' = \frac{g}{2} \int_{-\infty}^{+\infty} dk^1 \frac{k^1 |k^1|^{1/2}}{\sqrt{\pi E(k^1)}} [a^\dagger(k^1)c^\dagger(-k^1) + a(k^1)c(-k^1) - a^\dagger(k^1)c(k^1) - c^\dagger(k^1)a(k^1)], \quad (6)$$

i.e. a nondiagonal interaction Hamiltonian expressed in terms of

$$c(k^1) = \frac{i}{\sqrt{k^0}} \int dp^1 \{\theta(p^1 k^1) [b^\dagger(p^1)b(p^1 + k^1) - d^\dagger(p^1)d(p^1 + k^1)] + \\ + \epsilon(p^1)\theta(p^1(p^1 - k^1))d(k^1 - p^1)b(p^1)\}, \quad [c(k^1), c^\dagger(l^1)] = \delta(k^1 - l^1), \quad (7)$$

that define the bosonized current [8] $j^\mu(x) = -\frac{i}{\sqrt{2\pi}} \int \frac{dk^1}{\sqrt{2k^0}} k^\mu \{c(k^1)e^{-i\hat{k}\cdot x} - c^\dagger(k^1)e^{i\hat{k}\cdot x}\}$. Diagonalization of the full Hamiltonian by a suitable unitary operator $U = \exp(iS)$ would then generate also the new vacuum $|\Omega\rangle = N \exp[\gamma(g) \int_{-\infty}^{+\infty} dk^1 c^\dagger(-k^1) a^\dagger(k^1)] |0\rangle$. The two methods lead to a completely different vacuum structure of the RS model. The second (standard) treatment is incorrect. The same conclusions are valid also for the massive model with J^μ [7], where moreover the SL results fully agree with a parallel LF analysis.

3 The Thirring model

The operator solution of the Thirring model was given by Klaiber [8] who also calculated n-point correlation functions. Despite of having been studied intensively in the past, some of its properties have not been fully understood. Here we sketch a novel systematic Hamiltonian study based on the model's solvability. We start from the Lagrangian density

$$\mathcal{L} = \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - \frac{1}{2} g J_\mu J^\mu, \quad J^\mu = \bar{\Psi} \gamma^\mu \Psi, \quad \partial_\mu J^\mu(x) = 0. \quad (8)$$

The field equations are $i\gamma^\mu \partial_\mu \Psi(x) = g J^\mu(x) \gamma_\mu \Psi(x)$. The simplest solution is

$$\Psi(x) =: e^{-i(g/\sqrt{\pi})j(x)} : \psi(x), \quad \gamma^\mu \partial_\mu \Psi(x) = 0, \quad j_\mu(x) = \frac{1}{\sqrt{\pi}} \partial_\mu j(x), \quad J^\mu(x) = j^\mu(x). \quad (9)$$

$j(x)$ is the "integrated current". Free fields define the solution of the interacting model. The correct Hamiltonian is obtained by inserting the solution (9) to the Lagrangian. This reverses the sign of the interaction term also in the canonically obtained Hamiltonian $H = H_0 + H_g = \int_{-\infty}^{+\infty} dx^1 \left[-i\psi^\dagger \alpha^1 \partial_1 \psi - \frac{1}{2} g (j^0 j^0 - j^1 j^1) \right]$ (conventionally, one has $+\frac{1}{2}g$),

$$H = \int_{-\infty}^{+\infty} dk^1 |k^1| \left\{ b^\dagger(k^1) b(k^1) + d^\dagger(k^1) d(k^1) + \frac{g}{\pi} \left[c^\dagger(k^1) c^\dagger(-k^1) + c(k^1) c(-k^1) \right] \right\}. \quad (10)$$

To diagonalize H_g , we define the operator T with the same commutation property as H_0 , $T = \int_{-\infty}^{+\infty} dk^1 |k^1| c^\dagger(k^1) c(k^1)$, $[T, c(k^1)] = -|k^1| c(k^1)$, and the unitary operator $U = e^{iS}$,

$$S = -\frac{i}{2} \int_{-\infty}^{+\infty} dp^1 \gamma(p^1) [c^\dagger(p^1) c^\dagger(-p^1) - c(p^1) c(-p^1)]. \quad (11)$$

We form new Hamiltonians $\hat{H}_0 = H_0 - T$, $\hat{H}_g = H_g + T$. Due to $[S, \hat{H}_0] = 0$, \hat{H}_0 is invariant with respect to U , while \hat{H}_g transforms non-trivially, since $[S, c(k^1)] = i\gamma(k^1) c^\dagger(-k^1)$,

implying $c(k^1) \rightarrow e^{iS} c(k^1) e^{-iS} = c(k^1) \cosh \gamma(k^1) - c^\dagger(-k^1) \sinh \gamma(k^1)$. Consequently the interacting Hamiltonian acquires the most general operator form. It becomes diagonal,

$$\hat{H}_{int} = (\cosh 2\gamma_d)^{-1} \int_{-\infty}^{+\infty} dk^1 |k^1| c^\dagger(k^1) c(k^1), \quad (12)$$

if $\gamma(k^1) = \gamma_d = \frac{1}{2} \operatorname{arctanh} \frac{g}{\pi}$. Thus, we have achieved $e^{iS} \hat{H}_g e^{-iS} |0\rangle = 0$. This implies that the state $|\Omega\rangle = e^{-iS} |0\rangle$ is the true ground state of the original Hamiltonian. Explicitly,

$$|\Omega\rangle = \exp \left[-\frac{1}{2} \gamma_d \int_{-\infty}^{+\infty} dp^1 [c^\dagger(p^1) c^\dagger(-p^1) - c(p) c(-p^1)] \right] |0\rangle. \quad (13)$$

It is a coherent state of effective bosons, bilinear in fermion Fock operators. A direct computation shows that this state has zero momentum and axial charge. There is no spontaneous symmetry breaking, contrary to some statements in literature based on NJL type of approximations. The correlation functions, like for example $C_2(x-y) = \langle vac | \Psi(x) \bar{\Psi}(y) | vac \rangle$, should now be calculated from the normal-ordered operator solution (9) using an infrared cutoff and the vacuum state $|\Omega\rangle$. In the conventional treatment, one uses the Fock vacuum $|0\rangle$. Performing all necessary commutations, one finds

$$C_2(x-y) = e^{\frac{g^2}{\pi} D^{(+)}(x-y)} e^{-2g[D^{(+)}(x-y) + \gamma^5 \tilde{D}^{(+)}(x-y)]} \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle, \\ D^{(+)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk^1}{2|k^1|} \theta(|k^1| - \lambda) e^{-ik \cdot x} = -\frac{1}{4\pi} \ln(-\mu^2 x^2 + ix^0 \epsilon), \quad \mu = e^{\gamma_E} \lambda. \quad (14)$$

$\tilde{D}^+(x)$ has $\epsilon(k^1)$ in the integrand. We then find $\langle \Omega | \Psi(x) \bar{\Psi}(y) | \Omega \rangle = F_2(x-y; \kappa) C_2(x-y)$. $F_2(x-y; \kappa(g))$ is a complicated function equal to unity for $\kappa(g) = 0$, i.e. when $|\Omega\rangle \rightarrow |0\rangle$.

4 The Federbush model

The Federbush model (FM) [6] is the only known *massive* solvable model. Its true physical SL ground state can be found analogously to the massless Thirring model. This task requires a generalization of the Klaiber's bosonization to massive fermions. The model offers us a unique opportunity to solve a field theory nonperturbatively in both SL and LF forms and to compare their structures. The Lagrangian of the FM describes two species of the fermion field interacting via specific current-current coupling,

$$\mathcal{L} = \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m \bar{\Psi} \Psi + \frac{i}{2} \bar{\Phi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Phi - \mu \bar{\Phi} \Phi - g \epsilon_{\mu\nu} J^\mu H^\nu. \quad (15)$$

The currents are $J^\mu = \bar{\Psi} \gamma^\mu \Psi$, $H^\mu = \bar{\Phi} \gamma^\mu \Phi$. The coupled field equations read

$$i\gamma^\mu \partial_\mu \Psi(x) = m\Psi(x) + g\epsilon_{\mu\nu} \gamma^\mu H^\nu(x) \Psi(x), \quad i\gamma^\mu \partial_\mu \Phi(x) = \mu\Phi(x) - g\epsilon_{\mu\nu} \gamma^\mu J^\nu(x) \Phi(x). \quad (16)$$

The relations $J^\mu(x) = \epsilon^{\mu\nu} \partial_\nu j(x)/\sqrt{\pi}$, $H^\mu(x) = \epsilon^{\mu\nu} \partial_\nu h(x)/\sqrt{\pi}$ define the "integrated currents" $j(x)$ and $h(x)$. The latter enter into the solutions in an "off-diagonal" way:

$$\Psi(x) = e^{-i\frac{g}{\sqrt{\pi}}h(x)}\psi(x), \quad \Phi(x) = e^{i\frac{g}{\sqrt{\pi}}j(x)}\phi(x), \quad (17)$$

Here the free fields $\psi(x)$ and $\phi(x)$ are defined by $i\gamma^\mu \partial_\mu \psi(x) = m\psi(x)$, $i\gamma^\mu \partial_\mu \phi(x) = \mu\phi(x)$. The above solutions also imply $J^\mu(x) = j^\mu(x)$, $H^\mu(x) = h^\mu(x)$. Exponentials of the composite fields are more singular than in the massless case and have to be defined using the "triple-dot ordering" [9]. We avoid this by bosonization of the massive current.

The usual treatment yields contradictory picture of the dynamics of the model - the SL Hamiltonian contains interaction while the LF one has the free form:

$$\begin{aligned} H &= \int_{-\infty}^{+\infty} dx^1 \left[-\frac{i}{2} \psi^\dagger \alpha^1 \overset{\leftrightarrow}{\partial}_1 \psi + m \psi^\dagger \gamma^0 \psi - \frac{i}{2} \phi^\dagger \alpha^1 \overset{\leftrightarrow}{\partial}_1 \phi + \mu \phi^\dagger \gamma^0 \phi - g j^0 h^1 + g j^1 h^0 \right], \\ P^- &= \int_{-\infty}^{+\infty} \frac{dx^-}{2} \left[m (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + \mu (\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1) \right]. \end{aligned} \quad (18)$$

Our approach leads to a different picture. Inserting (17) into the Lagrangian, we get

$$\begin{aligned} \mathcal{L} &= \frac{i}{2} \psi^\dagger \gamma^0 \gamma^\mu \overset{\leftrightarrow}{\partial}_\mu \psi - m \bar{\psi} \psi + \frac{i}{2} \phi^\dagger \gamma^0 \gamma^\mu \overset{\leftrightarrow}{\partial}_\mu \phi - \mu \bar{\phi} \phi + g \epsilon_{\mu\nu} j^\mu h^\nu, \\ H &= \int_{-\infty}^{+\infty} dx^1 \left[-i \psi^\dagger \alpha^1 \partial_1 \psi + m \bar{\psi} \psi - i \phi^\dagger \alpha^1 \partial_1 \phi + \mu \bar{\phi} \phi + g(j^0 h^1 - j^1 h^0) \right]. \end{aligned} \quad (19)$$

Both operators have an opposite sign (with respect to the conventional result) in the interaction piece. The interaction term is non-diagonal when expressed in terms of bosonized massive currents. A Bogoliubov transformation is required to diagonalize it. The massive analogues of Klaiber's operators $c(k^1)$ are surprisingly complicated [10].

The LF form of the Lagrangian is

$$\begin{aligned} \mathcal{L}_{lf} &= i\Psi_2^\dagger \overset{\leftrightarrow}{\partial}_+ \Psi_2 + i\Psi_1^\dagger \overset{\leftrightarrow}{\partial}_- \Psi_1 - m(\Psi_2^\dagger \Psi_1 + \Psi_1^\dagger \Psi_2) + i\Phi_2^\dagger \overset{\leftrightarrow}{\partial}_+ \Phi_2 + i\Phi_1^\dagger \overset{\leftrightarrow}{\partial}_- \Phi_1 - \\ &\quad - \mu(\Phi_2^\dagger \Phi_1 + \Phi_1^\dagger \Phi_2) - \frac{g}{2} j^+ h^- + \frac{g}{2} j^- h^+. \end{aligned} \quad (20)$$

The field equations are

$$\begin{aligned} 2i\partial_+ \Psi_2(x) &= m\Psi_1 - gh^- \Psi_2, & 2i\partial_- \Psi_1 &= m\Psi_2 + gh^+ \Psi_1, \\ 2i\partial_+ \Phi_2(x) &= \mu\Phi_1 + gj^- \Phi_2, & 2i\partial_- \Phi_1 &= \mu\Phi_2 - gj^+ \Phi_1. \end{aligned} \quad (21)$$

The currents are $j^+(x) = 2 : \psi_2^\dagger(x) \psi_2(x) :$, $j^-(x) = 2 : \psi_1^\dagger(x) \psi_1(x) :$, $h^+(x) = 2 : \phi_2^\dagger(x) \phi_2(x) :$, $h^-(x) = 2 : \phi_1^\dagger(x) \phi_1(x) :$. Equations (21) are solved by (17) in terms of

the free LF fields and integrated currents. The bosonized form of the LF Hamiltonian is quadratic and diagonal:

$$P_g^- = \frac{g}{8\pi} \int_{-\infty}^{+\infty} dk^1 k^+ \left\{ [A^\dagger(k^+)D(k^+) - B^\dagger(k^+)C(k^+)] + H.c. \right\} \quad (22)$$

The operators $A(k^+)$, $B(k^+)$, $C(k^+)$ and $D(k^+)$ correspond to j^+ , h^+ , j^- and h^- . Their form is as simple as the massless $c(k^1)$ in the SL case. Complexities will enter in calculations of the correlation functions since the composite LF boson operators do not commute to the delta function at unequal LF times [10]. It will be interesting to analyze how the SL and LF schemes generate mutually consistent results for the correlators given the different vacuum structure in the two hamiltonian forms of the Federbush model.

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